INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2016; **26**:631–645
Published online 4 March 2015 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.3327

Necessary and sufficient conditions for bounded distributed mean square tracking of multi-agent systems with noises

Wuquan Li[1,2,3], Tao Li[4], Lihua Xie[2,*,†] and Ji-Feng Zhang[3]

[1]*School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China*
[2]*EXQUISITUS, Centre for E-City, School of Electrical and Electronic Engineering, Nanyang Technological University, Jurong 639798, Singapore*
[3]*Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[4]*Shanghai Key Laboratory of Power Station Automation Technology, School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China*

SUMMARY

This paper is concerned with the distributed control problem of second-order agents under directed network topology. The control input of each agent only depends on its own state and the states of its neighbors corrupted by white noises. By using the algebraic graph theory and stochastic analysis method, necessary and sufficient conditions are presented for mean square bounded tracking. Finally, several simulation examples are given to illustrate the results. Copyright © 2015 John Wiley & Sons, Ltd.

Received 25 April 2014; Revised 26 August 2014; Accepted 4 February 2015

KEY WORDS: distributed tracking; noises; leader-following; multi-agent systems

1. INTRODUCTION

In recent years, there has been an increasing research interest in distributed cooperative control of multi-agent systems. In particular, the so-called consensus control has been extensively studied. Consensus control generally means to design a network protocol, such that as time goes on, all agents asymptotically reach an agreement on their states to be coordinated. The consensus problem of first-order multi-agent systems has been studied in [1–3]. In [1], a systematic framework to analyze the first-order consensus algorithms is proposed. In [2], asymptotic information consensus under dynamically changing interaction topologies is realized for the case where the union of the directed interaction graphs has a spanning tree frequently enough as the system evolves. In [3], necessary and sufficient conditions are given on the consensus gains to achieve asymptotic unbiased mean square average consensus. Unlike the first-order case, [4] shows that having a (directed) spanning tree is a necessary rather than a sufficient condition for consensus with second-order dynamics. Compared with the first-order case, the second-order consensus problem is more complicated and challenging because all the states do not reach a consensus. The second-order consensus of multi-agent systems with a virtual leader in a dynamic proximity network is investigated in [5].

Recently, the leader-following consensus problem of multi-agent systems has received increasing attention [6–13], in which the leaders are usually independent of their followers, but have influence on the followers' behaviors. Therefore, one can realize one's control objective by only controlling the leader, which converts the control of the whole system into that of a single agent. Specifically, [6] considers the leader-following consensus problem of a group of autonomous agents with

*Correspondence to: Lihua Xie, Nanyang Technological University.
†E-mail: elhxie@ntu.edu.sg

time-varying coupling delays. In [7], the leader-follower problem for multi-agent systems with switching interconnection topologies is concerned. The leader-following consensus problem of second-order multi-agent systems with fixed and switching topologies as well as non-uniform time-varying delays is considered in [8]. When there exist noises in communication channels, [9] investigates the consensus of discrete-time leader-follower multi-agent systems and derives sufficient conditions guaranteeing the stochastic approximation type protocols with decreasing consensus gains to reach mean square consensus by using the stochastic Lyapunov analysis. In [10], the consensus of continuous-time leader-follower multi-agent systems is investigated and sufficient conditions for the mean square consensus are given by employing stochastic analysis and algebraic graph theory. In [11], the closed-loop control system is proved to be stochastically stable in the mean square sense by estimating the velocity of the active leader, and the sampled-data-based consensus tracking problem of second-order multi-agent systems is investigated in [12].

All of the aforementioned references assume that the agents can access the accelerations of its neighboring agents or its leader in their consensus algorithms. Recently, [13] relaxes this assumption and concentrates on the flocking problem using second-order tracking protocols in directed graphs with switching topology but does not consider the measurement noises. In this paper, we consider the distributed tracking of continuous-time second-order multi-agent systems with directed network topology. The control input of each agent can only depend on its own state and the states of its neighbors corrupted by stochastic communication noises. Compared with the existing work, the contributions of this paper include the following:

(i) This paper is the first to study the distributed tracking problem for continuous-time second-order leader-following multi-agent systems with noises. How to deal with the stochastic noises is non-trivial;
(ii) The leader and the followers are both second order. Also, the leader's acceleration cannot be accessed by the followers and stochastic communication noises are taken into account. To the best of our knowledge, there is no any result on this case;
(iii) Under a very general condition on the leader's acceleration, necessary and sufficient conditions are given for mean square bounded tracking. Furthermore, if the leader's acceleration has a limit when the time goes to infinity, we prove that the bound of the tracking error is tight.

The remainder of this paper is organized as follows. Section 2 offers some preliminary results. Section 3 describes the problem under investigation. Section 4 focuses on the mean square bounded tracking control of multi-agent systems. After that, in Section 5, several simulation examples are presented to show the effectiveness of the theoretical results. The paper is concluded in Section 6.

2. PRELIMINARIES

The following notation will be used throughout the paper. For a given vector or matrix X, X^T denotes its transpose and X^* its conjugate transpose. $\mathrm{Tr}\{X\}$ denotes its trace when X is square and $|X|$ is the Euclidean norm of a vector X. $\limsup_{t\to\infty} Y(t)$ is the superior limit of $Y(t)$. χ_M is the indicator function of M. R denotes the set of real numbers. $\mathbf{1_n} = (\underbrace{1, 1, \ldots, 1}_{n})^T$. E represents the mathematical expectation. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph of order n with the set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$, set of arcs $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A = (a_{ij})_{n\times n}$ with non-negative elements. $(j, i) \in \mathcal{E}$ means that agent j can directly send information to agent i; in this case, j is called the parent of i, and i is called the child of j. The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}, i \neq j\}$. $a_{ij} > 0$ if node j is a neighbor of node i and $a_{ij} = 0$ otherwise. Node i is called an isolated node, if it has neither parent nor child. Node i is called a source if it has no parents but children. Denote the sets of all sources and isolated nodes in $\mathcal{V}$ by $\mathcal{V}_s = \{j \in \mathcal{V} | N_j = \emptyset, \emptyset \text{ is the empty set}\}$. To avoid the trivial cases, $\mathcal{V} - \mathcal{V}_s \neq \emptyset$ is always assumed in this paper. A sequence $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ of edges is called a directed path from node i_1 to node i_k. A directed tree is a digraph, where every node except the root has exactly one parent and the root is the source. A spanning tree of $\mathcal{G}$ is a directed tree whose node set is $\mathcal{V}$ and whose

DOI: 10.1002/rnc

edge set is a subset of $\mathcal{E}$. A digraph $\mathcal{G}$ is strongly connected if there exists a path between any two distinct nodes. A strong component of a digraph is an induced subgraph that is maximal and strongly connected. The diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ is the degree matrix, whose diagonal elements $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$. The Laplacian of a weighted digraph $\mathcal{G}$ is defined as $L = D - A$.

We consider a system consisting of n agents and a leader (labeled by 0) that is depicted by a graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, where $\bar{\mathcal{V}} = \{0, 1, 2, \ldots, n\}$ and $\bar{\mathcal{E}} \subset \bar{\mathcal{V}} \times \bar{\mathcal{V}}$ is the set of arcs. If $(0, i) \in \bar{\mathcal{E}}$, then $0 \in \mathcal{N}_i$. A diagonal matrix $B = diag(b_1, b_2, \ldots, b_n)$ is the leader adjacency matrix associated with $\bar{\mathcal{G}}$, where $b_i > 0$ if node 0 is a neighbor of node i, and $b_i = 0$ otherwise.

The following lemmas will be used throughout the paper.

Lemma 1 ([6]*)*
All the eigenvalues of the matrix $H = L + B$ have positive real parts if and only if the leader is the root of a spanning tree in $\bar{\mathcal{G}}$.

Lemma 2 ([12]*)*
The directed graph $\mathcal{G}$ is strongly connected if and only if its Laplacian is irreducible.

3. PROBLEM FORMULATION

Consider the following second-order multi-agent system:

$$\dot{x}_i = v_i, \dot{v}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

where $x_i \in R$, $v_i \in R$, and $u_i \in R$ are the position, velocity, and the control input of agent i, respectively.

The leader for (1) is described by

$$\dot{x}_0 = v_0, \dot{v}_0 = a_0(t), \tag{2}$$

where $x_0 \in R$, $v_0 \in R$, and $a_0(t) \in R$ are the position, velocity, and acceleration of the leader, respectively.

For the acceleration $a_0(t)$ in (2), we make the following assumption that is reasonable.

Assumption 1
There exists an unknown constant $\bar{a}_0 \leqslant 0$ such that

$$|a_0(t)| \leqslant \bar{a}_0. \tag{3}$$

We also assume that the information received by the ith agent is corrupted by noises:

$$x_{ij} = \begin{cases} x_j + \sigma_{ij}\xi_{ij}, & j \in \mathcal{N}_i, \\ 0, & j \notin \mathcal{N}_i, \end{cases} \qquad x_{i0} = \begin{cases} x_0 + \sigma_{i0}\xi_{i0}, & 0 \in \mathcal{N}_i, \\ 0, & 0 \notin \mathcal{N}_i, \end{cases}$$

$$v_{ij} = \begin{cases} v_j + \rho_{ij}\xi_{ij}, & j \in \mathcal{N}_i, \\ 0, & j \notin \mathcal{N}_i, \end{cases} \qquad v_{i0} = \begin{cases} v_0 + \rho_{i0}\xi_{i0}, & 0 \in \mathcal{N}_i, \\ 0, & 0 \notin \mathcal{N}_i, \end{cases}$$

where $\{\xi_{ij}, i = 1, 2, \ldots, n, \quad j = 0, 1, 2, \ldots, n\}$ are mutually independent standard white noises, and $\sigma_{ij} \leqslant 0, \rho_{ij} \leqslant 0$ are noise intensities.

Let the distributed control law be given by

$$u_i = -k_1 \sum_{j=1}^{n} \big(a_{ij}(x_i - x_{ij}) + b_i(x_i - x_{i0})\big) - k_2 \sum_{j=1}^{n} \big(a_{ij}(v_i - v_{ij}) + b_i(v_i - v_{i0})\big), \tag{4}$$

where $k_1 > 0$ and $k_2 > 0$ are design parameters. Substituting (4) into (1) yields that

DOI: 10.1002/rnc

$$\dot{x}_i = v_i, \dot{v}_i = -k_1 \sum_{j=1}^{n} (a_{ij}(x_i - x_{ij}) + b_i(x_i - x_{i0})) - k_2 \sum_{j=1}^{n} (a_{ij}(v_i - v_{ij}) + b_i(v_i - v_{i0})), \quad (5)$$

where $i = 1, \ldots, n$.

By letting

$$\bar{x}(t) = (x_1, \ldots, x_n)^T - \mathbf{1_n} \otimes x_0, \\ \bar{v}(t) = (v_1, \ldots, v_n)^T - \mathbf{1_n} \otimes v_0, \quad (6)$$

one has that

$$\dot{\bar{x}}(t) = \bar{v}(t), \dot{\bar{v}}(t) = -k_1(L+B)\bar{x}(t) - k_2(L+B)\bar{v}(t) - \mathbf{1_n} \otimes a_0(t) + (k_1\sigma + k_2\rho)\,\xi, \quad (7)$$

where $\sigma = diag\left(\sum_{11}, \ldots, \sum_{1n}\right)$, $\sum_{1i} = (a_{i1}\sigma_{i1}, \ldots, a_{in}\sigma_{in}, b_i\sigma_{i0})$, $\rho = diag\left(\sum_{21}, \ldots, \sum_{2n}\right)$, $\sum_{2i} = (a_{i1}\rho_{i1}, \ldots, a_{in}\rho_{in}, b_i\rho_{i0})$, $\xi = (\xi_{11}, \ldots, \xi_{1n}, \xi_{10}, \ldots, \xi_{n1}, \ldots, \xi_{nn}, \xi_{n0})^T$.

With $\eta(t) = (\bar{x}^T(t), \bar{v}^T(t))^T$, we obtain that

$$\begin{aligned} d\eta(t) &= \begin{bmatrix} 0 & I_n \\ -k_1 H & -k_2 H \end{bmatrix} \eta(t)dt + \begin{bmatrix} 0 \\ -\mathbf{1_n} \otimes a_0(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ k_1\sigma + k_2\rho \end{bmatrix} d\omega \\ &= F\eta(t)dt + D(t)dt + Gd\omega, \end{aligned} \quad (8)$$

where $\omega = (\omega_{11}, \ldots, \omega_{1n}, \omega_{10}, \ldots, \omega_{n1}, \ldots, \omega_{nn}, \omega_{n0})^T$ is an $n(n+1)-$dimensional standard Brown motion, $H = L + B$.

Remark 1

For first-order agent systems like those studied in [3], because the position states converge invariably to a unique limit, time-varying consensus gains satisfying persistence conditions can be introduced in the consensus protocol to attenuate the noises. Unlike the first-order dynamics, as demonstrated by [4], second-order multi-agent systems have more complex dynamic structures, which makes the extension of consensus protocols from first-order to second-order challenging. For this reason, the persistence consensus gains are hardly useful to deal with the noises in the second order case, which can be evident later.

4. MEAN SQUARE BOUNDED TRACKING

Now, we give the definition of mean square bounded tracking.

Definition 1

The leader-following multi-agent system (1)–(2) with distributed control law (4) is said to achieve mean square bounded tracking if for system (8), there exists a constant $C > 0$ independent of t such that

$$\limsup_{t\to\infty} E|\eta(t)|^2 \leqslant C < +\infty.$$

Before proceeding to investigate the main results for the mean square bounded tracking problem under a fixed topology, we firstly establish the following lemmas that are essential to the derivation of the main results of this paper.

Lemma 3

For the F defined in (8), we have that F is asymptotically stable if and only if the following conditions hold:

DOI: 10.1002/rnc

(a) The leader is the root of a spanning tree in $\bar{\mathcal{G}}$;

(b) $\frac{k_2^2}{k_1} > \max\limits_{\beta \in \lambda(H)} \frac{|Im(\beta)|^2}{|\beta| Re(\beta)}$.

Proof

Firstly, we prove the necessity of (a). □

Let λ and β be, respectively, an eigenvalue of F and H; from the definition of F, one has

$$|\lambda I_{2n} - F| = \begin{vmatrix} \lambda I_n & -I_n \\ k_1 H & \lambda I_n + k_2 H \end{vmatrix} = |\lambda^2 I_n + k_2 \lambda H + k_1 H| = 0,$$

that yields

$$\lambda^2 + k_2 \beta \lambda + k_1 \beta = 0. \tag{9}$$

Let λ_1 and λ_2 be the roots of (9). Then, one has $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$. By Vieta's formulae, it holds that

$$-(\lambda_1 + \lambda_2) = k_2 \beta. \tag{10}$$

Noting that $k_2 > 0$ and (10), one obtains $Re(\beta) > 0$. By Lemma 1, the leader is the root of a spanning tree in $\bar{\mathcal{G}}$.

Then, by Hurwitz stability criteria, Lemma 2 of [14] and (a), the necessary and sufficient condition for $Re(\lambda) < 0$ is (b).

Remark 2

Condition (a) in Lemma 3 includes the following two aspects:

(i) There is a spanning tree in $\bar{\mathcal{G}}$;
(ii) The spanning tree originates from the leader. Specifically, all the followers can obtain the leader's information by information exchange with their neighbors.

Lemma 4

If F is Hurwitz, the solution of system

$$dx(t) = Fx(t)dt + Gd\omega. \tag{11}$$

satisfies

$$\lim_{t \to \infty} E|x(t)|^2 = Tr \left\{ \int_0^\infty e^{F\tau} G G^T e^{F^T \tau} d\tau \right\}. \tag{12}$$

Proof

Let $P(t) = E(x(t)x^T(t))$. Then, by Theorem 3.2 in [15], one has

$$\dot{P}(t) = FP(t) + P(t)F^T + GG^T, \tag{13}$$

from which and Corollary 1.1.6 in [16], noting that F is Hurwitz, one obtains

$$\lim_{t \to \infty} P(t) = \int_0^\infty e^{F\tau} G G^T e^{F^T \tau} d\tau. \tag{14}$$

By (14), it is easy to obtain (12). □

DOI: 10.1002/rnc

Lemma 5
Let λ_i, $i = 1, \ldots, s$, be numbers with negative real parts, $a_0(t)$ be the acceleration satisfying Assumption 1, $f(t,u) = \sum_{i=1}^{s}\sum_{k=0}^{n_i} \frac{d_{ik}}{k!} e^{\lambda_i(t-u)}(t-u)^k$, where d_{ik} is a constant, k and n_i are non-negative integers. Then one has

$$\limsup_{t\to\infty}\left|\int_{t_0}^{t} f(t,u)a_0(u)du\right| \leqslant \sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{|d_{ik}|\bar{a}_0}{(-Re(\lambda_i))^{k+1}}.$$

Furthermore, if $\lim\limits_{t\to\infty} a_0(t) = \hat{a}$, then one has

$$\lim_{t\to\infty}\left|\int_{t_0}^{t} f(t,u)a_0(u)du\right| = |\hat{a}|\left|\sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{d_{ik}}{(-\lambda_i)^{k+1}}\right|. \tag{15}$$

Proof
Let $A_i = Re(\lambda_i)$, $B_i = Im(\lambda_i)$, $A_i < 0$. By using the properties of Euler integral ($\gamma-$function), one obtains that

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{(t-u)^k}{k!}e^{\lambda_i(t-u)}du = \frac{1}{(-\lambda_i)^{k+1}}. \tag{16}$$

It follows from (16), the definition of $f(t,u)$ and Assumption 1 that

$$\begin{aligned}\left|\int_{t_0}^{t} f(t,u)a_0(u)du\right| &= \left|\sum_{i=1}^{s}\sum_{k=0}^{n_i}\int_{t_0}^{t}\frac{d_{ik}}{k!}e^{\lambda_i(t-u)}(t-u)^k a_0(u)du\right|\\ &\leqslant \sum_{i=1}^{s}\sum_{k=0}^{n_i}\int_{t_0}^{t}\frac{|d_{ik}|}{k!}e^{A_i(t-u)}(t-u)^k|a_0(u)|du\\ &\leqslant \sum_{i=1}^{s}\sum_{k=0}^{n_i}\bar{a}_0|d_{ik}|\int_{t_0}^{t}\frac{(t-u)^k}{k!}e^{A_i(t-u)}du,\end{aligned} \tag{17}$$

which together with (16) leads to that

$$\limsup_{t\to\infty}\left|\int_{t_0}^{t} f(t,u)a_0(u)du\right| \leqslant \sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{|d_{ik}|\bar{a}_0}{(-Re(\lambda_i))^{k+1}}. \tag{18}$$

If $\lim\limits_{t\to\infty} a_0(t) = \hat{a}$, by using (16) and L$'$Hospital rule

$$\lim_{t\to\infty}\int_{t_0}^{t} e^{\lambda_i(t-u)}\frac{(t-u)^k}{k!}a_0(u)du = \frac{\hat{a}}{(-\lambda_i)^{k+1}}. \tag{19}$$

By (19) and the definition of $f(t,u)$, (15) follows.
Based on Lemma 5, one can obtain the following result. □

Lemma 6
If F is Hurwitz, then

$$\limsup_{t\to\infty}\left(\int_{t_0}^{t} e^{F(t-u)}D(u)du\right)^*\left(\int_{t_0}^{t} e^{F(t-u)}D(u)du\right) \leqslant \sum_{l=1}^{2n}\left(\sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{|d_{lik}|\bar{a}_0}{(-Re(\lambda_i))^{k+1}}\right)^2. \tag{20}$$

DOI: 10.1002/rnc

In addition, if $\lim_{t\to\infty} a_0(t) = \hat{a}$, then one has

$$\lim_{t\to\infty}\left(\int_{t_0}^{t} e^{F(t-u)}D(u)du\right)^{*}\left(\int_{t_0}^{t} e^{F(t-u)}D(u)du\right) = \hat{a}^2\sum_{l=1}^{2n}\left|\sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{d_{lik}}{(-\lambda_i)^{k+1}}\right|^2, \tag{21}$$

where λ_i is an eigenvalue of F, n_i is the order of the Jordan block corresponding to λ_i, and d_{lik} $(l = 1,\ldots,2n, i = 1,\ldots,s, k = 0,\ldots,n_i)$ are constants determined by F.

Proof
For the matrix F, there exists an invertible matrix P such that

$$F = Pdiag(J_1,\ldots,J_s)P^{-1}, \tag{22}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \lambda_i & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{n_i}.$$

Then one has

$$e^{J_i(t-s)} = e^{\lambda_i(t-s)}\begin{bmatrix} 1 & t-s & \cdots & \frac{(t-s)^{n_i-1}}{(n_i-1)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & t-s \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{n_i},$$

that yields

$$\int_{t_0}^{t} e^{F(t-u)}D(u)du = -\left(\int_{t_0}^{t} A_1(t,u)a_0(u)du,\ldots,\int_{t_0}^{t} A_{2n}(t,u)a_0(u)du\right)^T, \tag{23}$$

where $A_l(t,u) = \sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{d_{lik}}{k!}e^{\lambda_i(t-u)}(t-u)^k$ with d_{lik} $(l = 1,\ldots,2n, i = 1,\ldots,s, k = 0,\ldots,n_i)$ being constants determined by F.

From (23), one has

$$\limsup_{t\to\infty}\left(\int_{t_0}^{t} e^{F(t-u)}D(u)du\right)^{*}\left(\int_{t_0}^{t} e^{F(t-u)}D(u)du\right) = \limsup_{t\to\infty}\sum_{l=1}^{2n}\left|\int_{t_0}^{t} A_l(t,u)a_0(u)du\right|^2. \tag{24}$$

By (24) and Lemma 5, it is easy to obtain (20) and (21). □

Lemma 7
For $H = L + B$, if H has a zero eigenvalue, then one can choose its corresponding left eigenvector $(x_1,\ldots,x_n)^T$ satisfying $x_n > 0$.

Proof
Because H has a zero eigenvalue, the leader is not the root of any spanning tree.

We prove this lemma from the following three cases:

1. If $\mathcal{G}$ has a spanning tree. Let $S_1,\ldots,S_p$ be the strong components of $\mathcal{G}$.

DOI: 10.1002/rnc

(a) If $p = 1$, $\mathcal{G}$ is strongly connected. Since the leader is not the root of any spanning tree, we have $B = 0$. From Lemma 1 in [17] and Lemma 2, one can choose the left eigenvector $(x_1, \ldots, x_n)^T$ corresponding to the zero eigenvalue satisfying $x_i > 0$, $i = 1, \ldots, n$. Noting that $H = L + B = L$, the conclusion follows.

(b) If $p > 1$, by Theorem 5 in [18], there exists a strong component S_2 with $L_{22} \in R^{r\times r}$ $(1 \leqslant r < n)$ as its Laplacian, having no neighbors. By rearranging the indices of n agents, one obtains

$$L = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}.$$

Since the leader is not the root of any spanning tree, $B = diag(B_{11}, B_{22})$ with $B_{22} = 0$. By Lemma 1 in [17] and Lemma 2, one can choose the left eigenvector corresponding to the zero eigenvalue as $(0, \ldots, 0, x_1, \ldots, x_r)^T$ satisfying $x_i > 0$, $i = 1, \ldots, r$. In view of the fact that

$$L + B = \begin{bmatrix} L_{11} + B_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix},$$

the result follows.

2. If $\mathcal{G}$ has no spanning tree and is weakly connected, then there must be two strong components without any links between them. By rearranging the indices of n agents properly, the Laplacian L can be written as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ 0 & L_{22} & 0 \\ 0 & 0 & L_{33} \end{bmatrix},$$

where L_{22} and L_{33} are the Laplacian matrix of the two strong components, $L_{12} \neq 0$ and $L_{13} \neq 0$. Let $B = diag(B_1, B_2, B_3)$. Since the leader is not the root of any spanning tree, we must have $B_2 = 0$ or $B_3 = 0$. Without loss of generality, we assume that $B_3 = 0$. Then similar to (b) of (1), we can conclude that the associated left eigenvector of the zero eigenvalue of H has its $x_n > 0$.

3. If $\mathcal{G}$ is not weakly connected, then by rearranging the indices of n agents properly, the Laplacian L can be written as

$$L = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix},$$

where L_{22} is the Laplacian matrix associated with a strong component. Denote $B = diag(B_1, B_2)$. Since H has a zero eigenvalue, we must have that $L_{11} + B_1$ or $L_{22} + B_2$ has a zero eigenvalue. If $L_{22} + B_2$ has a zero eigenvalue, since L_{22} is the Laplacian matrix associated with a strongly connected graph, then similar to (1), the result holds. If $L_{11} + B_1$ has a zero eigenvalue, then we may decompose L_{11} following the aforementioned procedure and finally obtain the conclusion of the lemma.

Based on Lemmas 3–7, in the following, we give the main results of this paper. □

Theorem 1

For any given $a_0(t)$ satisfying Assumption 1, the leader-following multi-agent systems (1)–(2), (4) can achieve mean square bounded tracking for any $(x_i(0), v_i(0))$ if and only if

1. The leader is the root of a spanning tree in $\bar{\mathcal{G}}$;
2. $\frac{k_2^2}{k_1} > \max\limits_{\beta\in\lambda(H)} \frac{|Im(\beta)|^2}{|\beta|Re(\beta)}$.

Furthermore, the tracking error satisfies

$$\limsup_{t\to\infty} E|\eta|^2 \leqslant \sum_{l=1}^{2n}\left(\sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{|d_{lik}|\bar{a}_0}{(-Re(\lambda_i))^{k+1}}\right)^2 + Tr\left\{\int_0^{\infty} e^{F\tau}GG^T e^{F^T\tau}d\tau\right\}. \tag{25}$$

In addition, if $\lim_{t\to\infty} a_0(t) = \hat{a}$ exists, then one has

$$\lim_{t\to\infty} E|\eta|^2 = \hat{a}^2\sum_{l=1}^{2n}\left|\sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{d_{lik}}{(-\lambda_i)^{k+1}}\right|^2 + Tr\left\{\int_0^{\infty} e^{F\tau}GG^T e^{F^T\tau}d\tau\right\}. \tag{26}$$

Proof
The necessity is proved from Lemma 3; we only need to prove that multi-agent systems (1)–(2), (4) can achieve mean square bounded tracking with (25) and (26) if and only if F is Hurwitz.

First, note that the solution of system (8) can be written as

$$\eta = e^{F(t-t_0)}\eta(t_0) + \int_{t_0}^{t} e^{F(t-u)}D(u)du + \int_{t_0}^{t} e^{F(t-u)}Gd\omega.$$

Sufficiency: Let

$$\begin{aligned}
\eta &= \eta_1 + \eta_2,\\
\eta_1 &= e^{F(t-t_0)}\eta(t_0) + \int_{t_0}^{t} e^{F(t-u)}Gd\omega,\\
\eta_2 &= \int_{t_0}^{t} e^{F(t-u)}D(u)du.
\end{aligned} \tag{27}$$

Since F is Hurwitz,

$$\lim_{t\to\infty} e^{F(t-t_0)}\eta(t_0) = 0. \tag{28}$$

By the definition of F and G, one has $E\{\int_{t_0}^{t} e^{F(t-u)}Gd\omega\} = 0$, which and (27)–(28) yields

$$\limsup_{t\to\infty} E\left(\eta_1^T\eta_2\right) = 0. \tag{29}$$

By combining Lemmas 4 and 6 and (29), one can obtain (25) and (26).

Necessity: Let

$$\begin{aligned}
\eta &= \eta_3 + \eta_4 + \eta_5,\\
\eta_3 &= e^{F(t-t_0)}\eta(t_0),\\
\eta_4 &= \int_{t_0}^{t} e^{F(t-u)}D(u)du,\\
\eta_5 &= \int_{t_0}^{t} e^{F(t-u)}Gd\omega.
\end{aligned} \tag{30}$$

Noting that $E\eta_5 = 0$, it is obvious that

$$E\left(\eta^T\eta\right) \leqslant \eta_3^T\eta_3 + 2\eta_3^T\eta_4. \tag{31}$$

DOI: 10.1002/rnc

Now, we prove when $Re(\lambda) \leqslant 0$, $\limsup_{t\to\infty} \left(\eta_3^T \eta_3 + 2\eta_3^T \eta_4\right) = +\infty$. For simplicity, we only consider the case of $\lambda = 0$. For the case of $Re(\lambda) > 0$ and $\lambda = qi, q \neq 0$, it follows similarly.

From (9), if $\lambda = 0$, one has $\beta = 0$. By substituting $\beta = 0$ into (9), F has at least two zero eigenvalues. We assume that F has two zero eigenvalues and the rest eigenvalues have negative real parts. With this assumption, from (9), H has only one zero eigenvalue. Therefore, the algebraic multiplicity and the geometric multiplicity of $\lambda_H = 0$ are both 1. Let $(\nu_1^T, \nu_2^T)^T$ be a left eigenvector corresponding to the zero eigenvalue for F, then one can obtain

$$\left(\nu_1^T, \nu_2^T\right) \begin{bmatrix} 0 & I_n \\ -k_1 H & -k_2 H \end{bmatrix} = \left(0^T, 0^T\right)^T, \tag{32}$$

which gives that $\nu_1 = 0, \nu_2^T H = 0$.

From (32) and the aforementioned discussions, it is easy to know that the algebraic multiplicity of $\lambda_F = 0$ is 2 and the geometric multiplicity of $\lambda_F = 0$ is 1. Therefore, the Jordan block for $\lambda_F = 0$ is

$$e^{J_s(t-t_0)} = \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix}. \tag{33}$$

Let $\eta(t_0) = (0, \ldots, 0, \alpha)$ with $\alpha \neq 0$ a constant to be chosen. Let p_{ij} be the (i, j)−th element of P^{-1}. By using (32), Lemma 7, and noting that P^{-1} is formed by the left eigenvectors and expanded left eigenvectors of F, one can obtain $p_{2n,2n} > 0$.

Thus,

$$\begin{aligned} \eta_3^T \eta_3 &= \eta^T(t_0) e^{F^T(t-t_0)} e^{F(t-t_0)} \eta(t_0) \\ &= \eta^T(t_0)(P^{-1})^T diag\left(e^{J_1^T(t-u)}, \ldots, e^{J_s^T(t-t_0)}\right) P^T P diag\left(e^{J_1(t-t_0)}, \ldots, e^{J_s(t-t_0)}\right) P^{-1}\eta(t_0) \\ &\geqslant \lambda_{min}(P^T P)\eta^T(t_0)(P^{-1})^T diag\left(e^{J_1^T(t-t_0)} e^{J_1(t-t_0)}, \ldots, e^{J_s^T(t-t_0)} e^{J_s(t-t_0)}\right) P^{-1}\eta(t_0), \end{aligned} \tag{34}$$

which yields

$$\begin{aligned} \lim_{t\to\infty} \eta_3^T \eta_3 &\geqslant \lambda_{min}(P^T P)\alpha^2 \left(p_{2n,2n}^2 (t - t_0)^2 + p_{2n-1,2n}^2 + 2p_{2n,2n} p_{2n-1,2n}(t - t_0) + p_{2n,2n}^2\right) \\ &= +\infty. \end{aligned} \tag{35}$$

Denote $\beta(t) = (0, \ldots, 0, 1)^T e^{F^T(t-t_0)} \int_{t_0}^t e^{F(t-s)} D(s) ds$ and choose α as follows:

(i) If $\limsup_{t\to\infty} \beta(t) = +\infty$, choosing α as any positive constant;
(ii) If $\limsup_{t\to\infty} \beta(t) = -\infty$, choosing α as any negative constant;
(iii) If $\limsup_{t\to\infty} \beta(t)$ is finite, choosing α as any non-zero constant.

Noting that

$$\eta_3^T \eta_4 = \alpha\beta(t), \tag{36}$$

with the definition of α and (9), we have

$$\begin{aligned} \limsup_{t\to\infty} \left(\eta_3^T \eta_3 + 2\eta_3^T \eta_4\right) &= \lim_{t\to\infty} \eta_3^T \eta_3 + 2\limsup_{t\to\infty} \eta_3^T \eta_4 \\ &\geqslant \lim_{t\to\infty} \eta_3^T \eta_3 \\ &= +\infty. \end{aligned} \tag{37}$$

DOI: 10.1002/rnc

It then follows from (31) and (37) that

$$\limsup_{t\to\infty} E\left(\eta^T\eta\right) = +\infty. \tag{38}$$

The necessity is proved. □

Remark 3
From (25), we can find that $\sum_{l=1}^{2n}\left(\sum_{i=1}^{s}\sum_{k=0}^{n_i}\frac{|d_{lik}|\bar{a}_0}{(-Re(\lambda_i))^{k+1}}\right)^2$ is produced by the acceleration of the leader and $Tr\left\{\int_0^\infty e^{F\tau}GG^Te^{F^T\tau}d\tau\right\}$ is produced by the noise intensity, which shows that the bound of the tracking error heavily relies on the acceleration of the leader and noise intensity.

Remark 4
When $G = 0$, the problem under investigation reduces to the case in [13]. However, in [13], only sufficient conditions are derived for bounded tracking. In Theorem 1, we present necessary and sufficient conditions for mean square bounded tracking.

It should be noted that in Theorem 1, we do not impose any extra assumption on the noise intensities $\sigma_{ij} \geqslant 0, \rho_{ij} \geqslant 0$. In particular, all the noise intensities can be zero simultaneously. In the following, we are interested in the case where some noise intensities are known to be non-zero.

We make the following assumption.

Assumption 2
In the union graph $\bar{\mathcal{G}}$, there exists at least an edge $(j,i) \in \mathcal{E}$ such that $\sigma_{ij} > 0$, or $\rho_{ij} > 0$, $j = 0, 1, 2, \ldots, n$.

Under Assumption 2, we can obtain the following result.

Theorem 2
For any given $a_0(t)$ satisfying Assumption 1, and any given initial condition $(x_i(0), v_i(0))$ and $(x_0(0), v_0(0))$, under Assumption 2, the leader-following multi-agent system (1)–(2), (4) can achieve mean square bounded tracking with the tracking error (25) and (26) if and only if

1. The leader is the root of a spanning tree in $\bar{\mathcal{G}}$;
2. $\frac{k_2^2}{k_1} > \max\limits_{\beta\in\lambda(H)}\frac{|Im(\beta)|^2}{|\beta|Re(\beta)}$.

Proof
The sufficiency is the same as that in the proof of Theorem 1. Now, we prove the necessity.

Let

$$\begin{aligned}
&\eta = \eta_6 + \eta_7,\\
&\eta_6 = e^{F(t-t_0)}\eta(t_0) + \int_{t_0}^{t} e^{F(t-u)}D(u)du,\\
&\eta_7 = \int_{t_0}^{t} e^{F(t-u)}Gd\omega.
\end{aligned} \tag{39}$$

Then, we have

$$E\left(\eta^T\eta\right) = E\left(\eta_6^T\eta_6\right) + E\left(\eta_7^T\eta_7\right) \geqslant E\left(\eta_7^T\eta_7\right). \tag{40}$$

In the following, we aim to prove that when $Re(\lambda) \geqslant 0$, $\lim\limits_{t\to\infty} E\left(\eta_7^T\eta_7\right) = +\infty$.

DOI: 10.1002/rnc

For λ and F, we make the same assumptions as that in the proof of the necessity of Theorem 1. By Itô isometry property of stochastic integral, noting that P^TP is positive definite and $Tr\{AB\} = Tr\{BA\}$, we have

$$\begin{aligned} E(\eta_7^T\eta_7) &= \int_{t_0}^{t} Tr\left\{e^{F(t-u)}GG^Te^{F^T(t-u)}\right\}du \\ &\geqslant \lambda_{\min}(P^TP)\int_{t_0}^{t} Tr\left\{diag\left(e^{J_1^T(t-u)}e^{J_1(t-u)},\dots,e^{J_s^T(t-u)}e^{J_s(t-u)}\right)P^{-1}GG^T(P^{-1})^T\right\}du, \end{aligned} \tag{41}$$

where $e^{J_s(t-u)}$ has the same structure with $e^{J_s(t-t_0)}$ in (33).

Let

$$P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{42}$$

where the dimensions of P_{11}, P_{12}, P_{21} and P_{22} are $(2n-1)\times(2n-1)$, $(2n-1)\times 1$, $1\times(2n-1)$, 1×1, respectively. Let G_1 be the $(2n-1)\times(2n-1)$'s order principal minor determinant of GG^T, and g_2 be the $(2n,2n)$th element of GG^T. Under Assumption 2, we know that $g_2 > 0$.

By the definition of G and (42), one has

$$\begin{aligned} P^{-1}GG^T(P^{-1})^T &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} G_1 & 0 \\ 0 & g_2 \end{bmatrix}\begin{bmatrix} P_{11}^T & P_{21}^T \\ P_{12}^T & P_{22} \end{bmatrix} \\ &= \begin{bmatrix} P_{11}G_1P_{11}^T + g_2P_{12}P_{12}^T & P_{11}G_1P_{21}^T + g_2P_{12}P_{22} \\ P_{21}G_1P_{11}^T + g_2P_{22}P_{12}^T & P_{21}G_1P_{21}^T + g_2P_{22}^2 \end{bmatrix}. \end{aligned} \tag{43}$$

From (32), with Lemma 7, one can obtain that $P_{22} > 0$. Therefore,

$$P_{21}G_1P_{21}^T + g_2P_{22}^2 \geqslant \lambda_{min}(G_1)P_{21}P_{21}^T + g_2P_{22}^2 > 0. \tag{44}$$

From (41), (43) and (44), it follows that

$$\begin{aligned} \lim_{t\to\infty} E(\eta_7^T\eta_7) &\geqslant \lim_{t\to\infty} \lambda_{\min}(P^TP)g_2P_{22}^2\int_{t_0}^{t}\left((t-u)^2+\alpha_1u+\alpha_2\right)du \\ &= +\infty, \end{aligned} \tag{45}$$

where α_1 and α_2 are constants.

By (40) and (45), the necessity is proved. □

Remark 5

It should be emphasized that Theorem 2 is not a special case of Theorem 1. Undoubtedly, under Assumption 2, Theorem 1 also holds. However, the result in Theorem 2 is more general than that in Theorem 1. Specifically, Theorem 2 holds for more general initial condition $(x_i(0), v_i(0))$ and $(x_0(0), v_0(0))$.

From Theorems 1 and 2, one immediately obtains the following result:

Corollary 1

If the acceleration $a_0(t)$ satisfying $\lim_{t\to\infty} a_0(t) = 0$, with the noisy intensity $G = 0$, $\lim_{t\to\infty} E|\eta(t)|^2 = 0$.

DOI: 10.1002/rnc

5. NUMERICAL SIMULATION EXAMPLES

Example 1

We consider a system consisting of one leader and three followers. The communication topology of three followers is described by the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{E} = \{(1, 2)\}$, and $A = (a_{ij})_{3\times3}$ with $a_{21} = 1$, $a_{11} = a_{12} = a_{13} = a_{22} = a_{23} = a_{31} = a_{32} = a_{33} = 0$. The communications between the leader and the three followers can be described by $b_1 = b_3 = 1$, $b_2 = 0$. Obviously, the leader is globally reachable by the three followers. The noise intensity is chosen as $\sigma_{21} = \rho_{21} = \sigma_{10} = \rho_{10} = \sigma_{30} = \rho_{30} = 1$ and the acceleration $a(t) = \sin t$, by choosing $k_1 = k_2 = 1$, the conditions of Theorem 1 are satisfied. By choosing $x_0(0) = -1$, $v_0(0) = -1$, $x_1(0) = 1$, $v_1(0) = -0.5$, $x_2(0) = -3$, $v_2(0) = -0.5$, $x_3(0) = 5$, $v_3(0) = -0.5$, the trajectories of the position and velocity of the system are given in Figures 1 and 2, respectively, from which one can find that the leader-following multi-agent system achieved mean square bounded tracking.

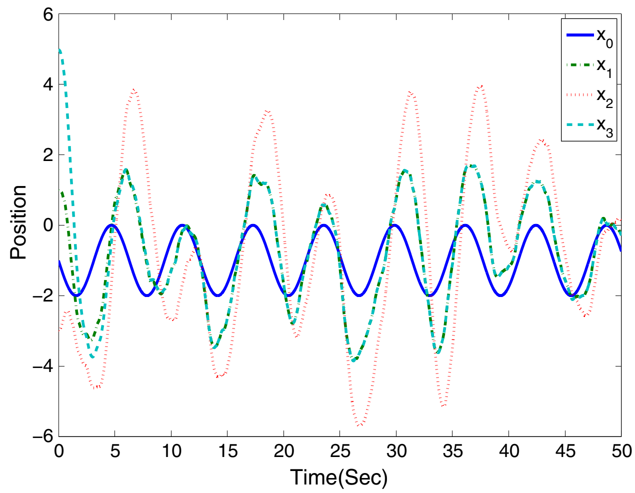

Figure 1. The trajectories of the positions with noise and acceleration.

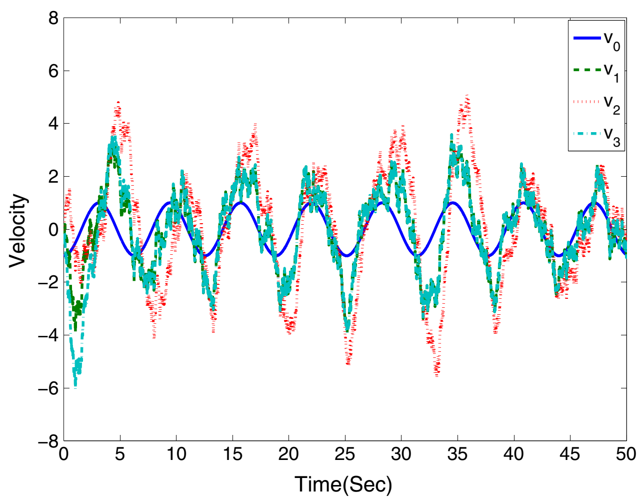

Figure 2. The trajectories of the velocities with noise and acceleration.

DOI: 10.1002/rnc

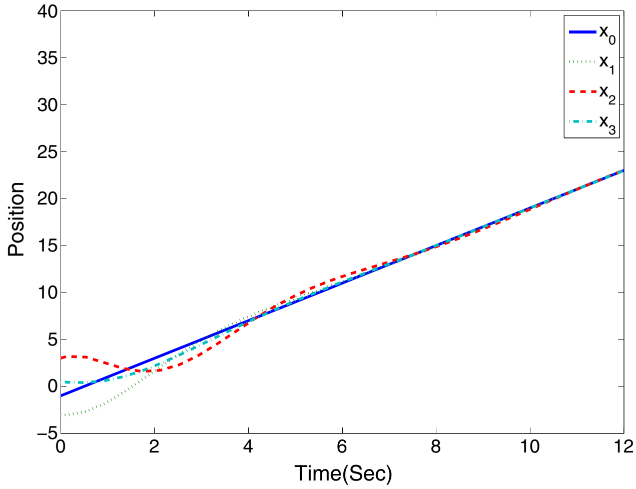

Figure 3. The trajectories of the positions without noise and acceleration.

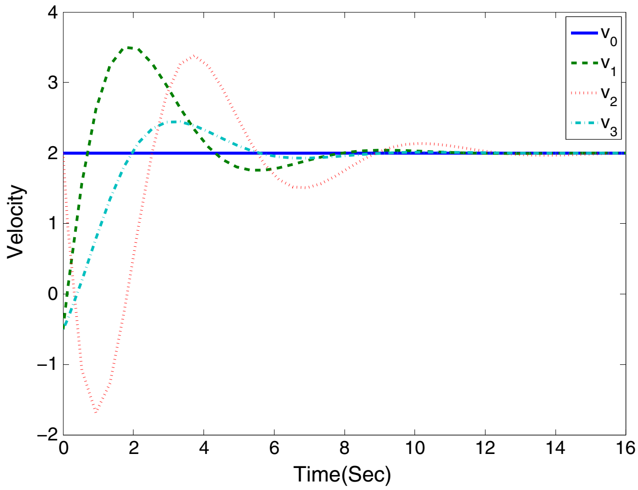

Figure 4. The trajectories of the velocities without noise and acceleration.

Example 2

Consider the same system as in Example 1. Suppose that all the noise intensities are zero and the acceleration of the leader is zero. By choosing $k_1 = k_2 = 1$, $x_0(0) = -1$, $v_0(0) = 2$, $x_1(0) = -3$, $v_1(0) = -0.5$, $x_2(0) = 3$, $v_2(0) = 2$, $x_3(0) = 0.5$, $v_3(0) = -0.5$, the trajectories of the position and velocity of the system are given in Figures 3 and 4, respectively, from which one can find that the leader-following multi-agent system achieved asymptotic tracking.

6. CONCLUSIONS

In this paper, we have investigated the distributed tracking problem of leader-follower multi-agent systems with measurement noises under directed topology. By using stochastic analysis tools, some necessary and sufficient conditions have been obtained for mean square bounded tracking.

There are many related problems to be investigated. For example, when there are delays in the communication channels, how to design a controller for tracking, and how to generalize the results of this paper to the cases of multi-agent systems with measurement noises based on sampled-data control.

DOI: 10.1002/rnc

ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation of China under grant nos. 61104128, 61370030, and 61120106011; the National Key Basic Research Program of China (973 program) under grant no. 2014CB845301; Shanghai Rising-Star Program under grant 15QA1402000; the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning; Promotive research fund for excellent young and middle-aged scientists of Shandong Province under Grant BS2013DX001.

REFERENCES

1. Olfati-Saber R, Murray R. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* 2004; **49**:1520–1533.
2. Ren W, Beard R. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control* 2005; **50**:655–661.
3. Li T, Zhang JF. Mean square average-consensus under measurement noises and fixed topologies: necessary and sufficient conditions. *Automatica* 2009; **45**:1929–1936.
4. Ren W, Atkins E. Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control* 2007; **17**:1002–1033.
5. Su H, Chen G, Wang X, Lin Z. Adaptive second-order consensus of networked mobile agents with nonlinear dynamics. *Automatica* 2011; **47**:368–375.
6. Hu JP, Hong YG. Leader-following coordination of multi-agent systems with coupling time delays. *Physica A* 2007; **374**:853–863.
7. Hong YG, Chen GR, Bushnellc LD. Distributed observers design for leader-following control of multi-agent networks. *Automatica* 2008; **44**:846–850.
8. Zhu W, Cheng DZ. Leader-following consensus of second-order agents with multiple time-varying delays. *Automatica* 2010; **46**:1994–1999.
9. Huang M, Manton JH. Coordination and con- sensus of networked agents with noisy measurements: stochastic algorithms and asymptotic behavior. *SIAM Journal on Control and Optimization* 2009; **48**:134–161.
10. Ma CQ, Li T, Zhang JF. Consensus control for leader-following multi-agent systems with measurement noises. *Journal of Systems Science and Complexity* 2010; **23**:35–49.
11. Hu JP, Feng G. Distributed tracking control of leader-follower multi-agent systems under noisy measurement. *Automatica* 2010; **46**:1382–1387.
12. Wu ZH, Peng L, Xie LB, Wen JW. Stochastic bounded consensus tracking of second-order multi-agent systems with measurement noises and sampled-data. *Automatica* 2012; **68**:261–273.
13. Guo WL, Lü JH, Chen SH, Yu XH. Second-order tracking control for leader-follower multi-agent flocking in directed graphs with switching topology. *Systems & Control Letters* 2011; **60**:1051–1058.
14. Zhu JD, Tian YP, Kuang J. On the general consensus protocol of multi-agent systems with double-integrator dynamics. *Linear Algebra and its Applications* 2009; **431**:701–715.
15. Mao XR. *Stochastic Differential Equations and Applications*: Horwood, 1997.
16. Abou-Kandil H, Freiling G, Ionesco V, Jank G. *Matrix Riccati Equations in Control and Systems Theory*. Birkhäuser Verlag: Basel, 2003.
17. Lu W, Chen T. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D* 2006; **213**(2):214–230.
18. Moreau L. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control* 2005; **50**(2):169–182.